KEY

Name:

ID number:

Signature:

Useful information:

Permittivity of vacuum: $\varepsilon_o = 8.854187816 \times 10^{-12}$ $C^2 \cdot J^{-1} \cdot m^{-1}$

$4\pi\varepsilon_o = 1.112650056 \times 10^{-10}$ $C^2 \cdot J^{-1} \cdot m^{-1}$

Bohr magneton: $\beta = 9.2740154 \times 10^{-24}$ $J \cdot T^{-1}$

Planck constant: $h = 6.6260755 \times 10^{-34}$ J.s

Proton charge: $e = 1.60217733 \times 10^{-19}$ C

Speed of light in vacuum: $c = 2.99792458 \times 10^{8}$ $m.s^{-1}$

Electron rest mass: $m_e = 9.1093897 \times 10^{-31}$ kg

Bohr radius: $a_o = 5.29177249 \times 10^{-11}$ m

$$\Psi(x) = \left(\frac{2}{L}\right)^{1/2} \sin\frac{n\pi x}{L} \text{ for a particle in a box} \qquad E_n = \frac{n^2 h^2}{8mL^2} \text{ for a particle in a box}$$

$$\sin x \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y) \qquad E_n = \frac{-m_e e^4}{32\pi^2 \varepsilon_o^2 \hbar^2 n^2} = \frac{-e^2}{8\pi\varepsilon_o a_o n^2} \text{ or } \frac{-e^2}{2a_o n^2} \text{ for H atom}$$

O is hermitian: $\langle m|O|n\rangle = \langle n|O|m\rangle^{*}$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$
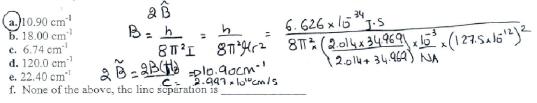
$$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\Psi_o(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}; \quad \Psi_1(x) = \left(\frac{4\alpha^3}{\pi}\right)^{1/4} x e^{-\alpha x^2/2} \text{ for 1D harmonic oscillator}; \quad \alpha = \frac{(k\mu)^{1/2}}{\hbar}$$

$$\int_0^\infty x^{2n} e^{-\alpha x^2} dx = \frac{1.3.5...(2n-1)}{2^{n+1}}\sqrt{\frac{\pi}{\alpha^{2n+1}}} \qquad \int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}; \text{ n is a positive integer}$$

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Harmonic oscillator energy: $E_\upsilon = \left(\upsilon + \frac{1}{2}\right)h\nu$; $\upsilon = 0, 1, 2, 3....$ $\nu = \frac{1}{2\pi}\left(\frac{k}{\mu}\right)^{1/2}$

Rigid rotor energy: $E_j = \frac{\hbar^2}{2I}J(J+1)$; $J = 0, 1, 2, 3...$

Part I. (24%) Instructions: Circle one answer. No penalty. 4 points each.

• Assuming an internuclear distance of 127.5 pm in $D^{35}Cl$, compute the expected line separation in cm^{-1} for the absorption peaks corresponding to absorption from $J=0 \to J=1$. (atomic masses: D = 2.0141 a.m.u, ^{35}Cl = 34.9688 a.m.u). $\mu = \frac{m_1 m_2}{m_1 + m_2}$

a. 10.90 cm^{-1}
b. 18.00 cm^{-1}
c. 6.74 cm^{-1}
d. 120.0 cm^{-1}
e. 22.40 cm^{-1}
f. None of the above, the line separation is _____

$2\hat{B}$

$B = \frac{h}{8\pi^2 I} = \frac{h}{8\pi^2 \mu r^2} = \frac{6.626 \times 10^{-34} J\cdot s}{8\pi^2 \times \left(\frac{2.014 \times 34.969}{2.014 + 34.969}\right) \times \frac{10^{-3}}{N_A} \times \left(127.5 \times 10^{-12}\right)^2}$

$2\tilde{B} = \frac{2B(h)}{c} = 10.90cm^{-1}$ $c = 2.997 \times 10^{10} cm/s$

• The Spherical Harmonics, wavefunction solutions of the Rigid Rotor, are eigenfunctions of which of the following operators:

a. **H**, the Hamiltonian of the H-atom
b. **L^2**, where **L** is the angular momentum operator
c. **L$_z$**, the angular momentum operator along the z-direction
d. **L^2/2I**, where **L** is the angular momentum operator, and I is the moment of inertia
e. All of the above
f. b), c), and d)

→ see back

• The IR-vibrational spectrum of $H^{81}Br$ consists of one intense line at 2461 cm^{-1}. The force constant of the bond between hydrogen and bromine is:

$\tilde{\nu} = 2461 cm^{-1}$ $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{89 \times 10^{-3}}{90 \times N_A} = 1.642 \times 10^{-27} kg$

a. 478 N.m^{-1}
b. 291 N.m^{-1}
c. 512 N.m^{-1}
d. 406 N.m^{-1}
e. None of the above. The answer is _____

$\nu_{s^{-1}} = c\tilde{\nu} = \frac{1}{2\pi}\left(\frac{k}{\mu}\right)^{1/2} \Rightarrow k = \mu c^2 \tilde{\nu}^2 4\pi^2 = 1.642 \times 10^{-27} kg \times$
$\left(2.9979 \times 10^8 ms^{-1}\right) \times \left(2461 cm^{-1} \times 10^2 cm/m\right)^2 \times 4\pi^2 = 406 Nm^{-1}$

• Which of the following functions is an "unacceptable" wavefunction to describe a system over the indicated interval.

a. $sin^{-1}(x)$; x (-1, 1) → unacceptable, multivalued function $sin 1 = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi$ ---
b. $exp(-x^2)$; x (-∞,∞)
c. $exp(-\beta x)$, where β is a negative real number; x (-∞,∞) → unacceptable, as x → ∞, $e^{-\beta x} \to \infty$
d. $exp(-r^2)$; r (0, ∞)
e. a) and c)

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• Consider the following wavefunction, which is a solution to the Schrödinger equation for the H-atom:

$$\Psi = \left(\frac{1}{a_o}\right)^{3/2}\left(27 - \frac{18r}{a_o} + \frac{2r^2}{a_o^2}\right)\exp(-r/3a_o)$$

This function corresponds to which of the following orbitals? (*Hint: you should be able to do this by inspection*)

2 radial nodes, o angular nodes
⟹ n = 3   l = 0 ⟹ 3s

a.   2s
b.   3s
c.   3p$_x$
d.   4p$_y$
e.   4s

also you can know n from exp -r/3 a$_o$

• Consider a linear polyene molecule (with 6π electrons) of length 0.95 nm (assumed linear). Calculate the wavelength of light required to promote an electron from the 3rd to the 4th energy level.

$$\Delta E = \frac{hc}{\lambda} = \frac{h^2}{8mL^2}(16-9) \Rightarrow \lambda = \frac{8mL^2c}{h(16-9)} = \frac{8 \times 9.1093879 \times 10^{-3} kg}{6.6266755 \times 10^{-34} J \cdot s} \times$$

a.   495 nm
b.   460 nm
c.   595 nm
d.   425 nm
e.   None of the above. The answer is _____

$$\frac{(0.95 \times 10^{-9} m)^2 \times 2.9979 \times 10^8 ms^{-1}}{(16-9)} = 4.25 \times 10^{-7} m = 425 nm$$

Part II. (20%)   Instructions: Circle T for True or F for False, **2 points each, no penalty.**

If a wavefunction Ψ is a linear combination of several eigenfunctions (Φ_n) of the operator $\hat{A}$ (corresponding to an observable A) then each measurement of the property A for a system in a state described by this wavefunction Ψ will be an eigenvalue of an eigenfunction (Φ_n) in the linear combination.   **(T)**   F

The zero point energy for a harmonic oscillator is zero.   T   **(F)**

Microscopic particles *that are confined in space* exhibit quantization of their eigenspectrum (quantization of energy levels)   **(T)**   F

If a system is in a state described by the wavefunction ψ, which is an eigenfunction of operator $\hat{A}$ then a measurement on the system in this state of the observable A can give any value as long as the average of all measurements is the eigenvalue a ($\hat{A}\psi = a\psi$).   T   **(F)**

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The spacing of the electronic energy levels under the potential energy of the H-atom decreases with increasing principal quantum number n, while the energy spacing for a quantum mechanical harmonic oscillator increases as the quantum number v increases.
T (F)

A particle according to quantum mechanics can exist in a region where its kinetic energy would be negative, whereas classically the particle cannot be present in such a region.
(T) F

An increase in particle mass would decrease the ground state energy of both a quantum particle-in-a-box and a quantum harmonic oscillator.
(T) F

The Gross selection rule for a molecule to have a rotational absorption spectrum is that the molecule has a permanent dipole moment.
(T) F

A free microscopic particle *confined in space* must have a non-zero (finite) zero-point energy, otherwise the system will be in defiance of the Heisenberg Uncertainty principle.
(T) F

The kinetic energy and the potential energy for an electron in an H-atom orbital can be determined simultaneously to arbitrarily high precision since each orbital wavefunction is an eigenfunction of the Hamiltonian operator, and therefore the electron in that orbital has a constant total energy (the eigenvalue).
T (F)

Part III. (56 %)

III.1 (6 %) For each of the following operators indicate by "yes" or "no" whether $\Psi_{(2P_y)}$ (with z =1; H-atom) is an eigenfunction. If it is, then also give the eigenvalue.

a. $-(\hbar^2/2m_e)\nabla^2 -e^2/r$ ①

yes

$E_{n=2} = \dfrac{-e^2}{8a_0}$ ① $\left(\text{or } \dfrac{-e^2}{32\pi\varepsilon_0 a_0} \right)$

b. $\hat{L}_z$ ②

No

$2P_y : \text{linear combination of}$
$\quad \psi_{211} \quad \psi_{21-1}$
$\quad ml=+1 \quad ml=-1$

c. $\hat{L}^2$ ①

Yes

$\ell(\ell+1)\hbar^2 = \overset{①}{1}(1+1)\hbar^2 = 2\hbar^2$
$L^2 = 2\hbar^2$

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III.2 (11 %) Consider the 2s electron of the H atom. Prove that the expectation value of the potential energy <V> = 2 <E>, where <E> is the expectation value of the total energy.

$$\Psi_{200} = \frac{1}{\sqrt{32\pi}}\left(\frac{1}{a_o}\right)^{3/2}\left(2-\frac{r}{a_o}\right)\exp(-r/2a_o)$$

$$\langle E\rangle = E_2 = -\frac{e^2}{8a_0} \quad \left(\text{or} \quad -\frac{e^2}{8\pi\varepsilon_o a_0 \cdot 4} = -\frac{e^2}{32\pi\varepsilon_o a_0}\right) \quad : \text{the expectation value of energy is the eigenvalue } E_{n=2}$$

b $\langle V\rangle = \int_{\text{all space}} \Psi_{200}^{*} \, V \, \Psi_{200} \, d\tau$
$\quad\quad V = \frac{-e^2}{4\pi\varepsilon_o} \quad$ or $V = -\frac{e^2}{r}$. take $V = -\frac{e^2}{r}$

$\langle V\rangle = \int_{0}^{\infty} \Psi_{200}^{2}\left(\frac{-e^2}{r}\right)\cdot 4\pi r^2 \, dr = -\frac{4\pi e^2}{32\pi a_0^3}\int_{0}^{\infty}\left(2-\frac{r}{a_o}\right)^2 r \, \exp\frac{-r}{a_o} \, dr$

$-\frac{e^2}{8a_0^3}\int_{0}^{\infty}\left(4+\frac{r^2}{a_o^2}-\frac{4r}{a_o}\right)r \, \exp\frac{-r}{a_o} \, dr = \frac{-e^2}{8a_0^3}\int_{0}^{\infty}\left(4r+\frac{r^3}{a_o^2}-\frac{4r^2}{a_o}\right)\exp\frac{-r}{a_o} \, dr$

$= -\frac{e^2}{8a_0^3}\left[4\cdot\frac{1}{\left(\frac{1}{a_o}\right)^2} + \frac{1}{a_o^2}\frac{3!}{\left(\frac{1}{a_o}\right)^4} - \frac{4}{a_o}\frac{2!}{\left(\frac{1}{a_o}\right)^3}\right]$

$= -\frac{e^2}{8a_0^3}\left[4a_0^2 + 6a_0^2 - 8a_0^2\right] = -\frac{e^2}{4a_0} = 2\left(-\frac{e^2}{8a_0}\right) = 2\langle E\rangle$

or $\langle V\rangle = -\frac{e^2}{16\pi\varepsilon_o a_0}$
$\langle E\rangle = E_2 = -\frac{e^2}{32\pi\varepsilon_o a_0}$ $\Bigg\}$ $\Rightarrow \langle V\rangle = 2\langle E\rangle$

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III.3 (12 %) Consider the Hydrogen atom in the 1s state:

$$\Psi_{100} = \frac{1}{\sqrt{\pi}} \left(\frac{1}{a_o} \right)^{3/2} \exp(-r/a_o)$$

a. Calculate the value of r at the classical turning point (in terms of a_o).

the classical turning point occurs (when $E < V$: classically forbidden region)

$$\Downarrow$$

$E = V$ is turning point

$$\frac{-e^2}{2a_o} = -\frac{e^2}{r} \Rightarrow \boxed{r = 2a_o} \text{ classical turning point}$$

at $r > 2a_o$, classically forbidden region

b. Calculate the percentage of electric charge present beyond the classical turning point.

% electric charge beyond the classical turning point = probability of finding an electron beyond $r = 2a_o$ (i.e, $r > 2a_o$).

$$= \int_{2a_o}^{\infty} \Psi_{100}^* \Psi_{100} \, d\tau = 1 - \int_0^{2a_o} \Psi_{100}^* \Psi_{100} \, d\tau = 1 - \int_0^{2a_o} \Psi_{100}^2 \, d\tau$$

$$\int_0^{2a_o} \Psi_{100}^2 \, d\tau = \frac{4\pi}{\pi} \cdot \frac{1}{a_o^3} \cdot \int_0^{2a_o} r^2 \exp -\frac{2r}{a_o} \, dr = \frac{4}{a_o^3} \int_0^{2a_o} \left(-\frac{a_o}{2} \right) r^2 \exp -\frac{2r}{a_o} \cdot d -\frac{2r}{a_o}$$

$$= \frac{4}{a_o^3} \left(-\frac{a_o}{2} \right) \left[r^2 \exp -\frac{2r}{a_o} \Big|_0^{2a_o} - \int_0^{2a_o} \exp -\frac{2r}{a_o} \cdot dr^2 \right] =$$

$$= -\frac{2}{a_o^2} \left[r^2 \exp -\frac{2r}{a_o} \Big|_0^{2a_o} - \int_0^{2a_o} 2r \cdot \left(-\frac{a_o}{2} \right) \cdot \exp -\frac{2r}{a_o} \cdot d -\frac{2r}{a_o} \right]$$

$$= -\frac{2}{a_o^2} \left[r^2 \exp -\frac{2r}{a_o} \Big|_0^{2a_o} + a_o \left(r \exp -\frac{2r}{a_o} \Big|_0^{2a_o} - \int_0^{2a_o} \exp -\frac{2r}{a_o} \, dr \right) \right]$$

$$= -\frac{2}{a_o^2} \left[r^2 \exp -\frac{2r}{a_o} \Big|_0^{2a_o} + a_o r \exp -\frac{2r}{a_o} \Big|_0^{2a_o} - a_o \cdot \left(-\frac{a_o}{2} \right) \cdot \exp -\frac{2r}{a_o} \Big|_0^{2a_o} \right]$$

$$= -\frac{2}{a_o^2} \left(4a_o^2 e^{-4} + 2a_o^2 e^{-4} + \frac{a_o^2}{2} e^{-4} - \frac{a_o^2}{2} \right) = -8e^{-4} - 4e^{-4} - e^{-4} + 1 \quad 6$$

$$= 1 - 13 e^{-4}. \Rightarrow \int_{2a_o}^{\infty} \Psi_{100}^2 \, d\tau = 1 - (1 - 13 e^{-4}) = 13 e^{-4} = 0.238$$

$$\Rightarrow 23.8\% \text{ electric charge } (e).$$

III.4 (12 %) Consider a wavefunction $\Psi(x,t)$ for a one-dimensional system that is the superposition of two stationary-state wavefunctions with energies E_1 and E_2, i.e.,:

$$\Psi(x,t) = a\exp(-iE_1 t/\hbar)\Psi_1(x) + b\exp(-iE_2 t/\hbar)\Psi_2(x)$$

where $\Psi_1(x)$ and $\Psi_2(x)$ are real orthonormal spatial eigenfunctions that obey the relations:

$$\int \Psi_1(x)^2 dx = \int \Psi_2(x)^2 dx = 1; \int \Psi_1(x)\Psi_2(x)dx = 0.$$

(a) Show that for $\Psi(x,t)$ to be normalized the constants a and b must obey $a^2 + b^2 = 1$

$\int \Psi(x,t)^* \Psi(x,t) dx = \int [a\exp iE_1 t/\hbar\, \Psi_1(x) + b\exp iE_2 t/\hbar\, \Psi_2(x)][a\exp -iE_1 t/\hbar\, \Psi_1(x)$

$+ b\exp -iE_2 t/\hbar\, \Psi_2(x)] dx = \int (a^2\Psi_1^2(x) + ab \exp iE_1 t/\hbar \cdot \exp -iE_2 t/\hbar\, \Psi_1(x)\cdot\Psi_2(x) +$

$ba \exp iE_2 t/\hbar \cdot \exp -iE_1 t/\hbar\, \Psi_2(x)\cdot\Psi_1(x) + b^2\Psi_2^2(x)) dx$

$a^2 \int \Psi_1^2 dx + ab \exp \frac{it}{\hbar}(E_1-E_2) \int \Psi_1\Psi_2 dx + ba \exp \frac{it}{\hbar}(E_2-E_1) \int \Psi_1\Psi_2 dx$

$+ b^2 \int \Psi_2^2 dx = a^2(1) + 0 + 0 + b^2(1) = a^2 + b^2 \Rightarrow a^2 + b^2 = 1$

(b) Show that the probability $|\Psi(x,t)|^2$ varies in time according to:

$$|\Psi(x,t)|^2 = a^2\Psi_1(x)^2 + b^2\Psi_2(x)^2 + 2ab\Psi_1(x)\Psi_2(x)\cos[(E_1-E_2)t/\hbar]$$

$|\Psi(x,t)|^2 = \Psi(x,t)^* \Psi(x,t) = a^2\Psi_1^2 + ab\,\Psi_1\Psi_2 \exp\frac{it}{\hbar}(E_1-E_2) + ba\,\Psi_2\Psi_1\exp\frac{it}{\hbar}(E_2-E_1)$

$\cdot\, b^2\Psi_2(x) = a^2\Psi_1(x)^2 + b^2\Psi_2(x)^2 + ab\,\Psi_1(x)\Psi_2(x)[\exp\frac{it}{\hbar}(E_1-E_2) + \exp\frac{-it}{\hbar}(E_1-E_2)]$

$\cdot\, a^2\Psi_1(x)^2 + b^2\Psi_2(x)^2 + ab\,\Psi_1(x)\cdot\Psi_2(x)[\cos\frac{t}{\hbar}(E_1-E_2) + i\sin\frac{t}{\hbar}(E_1-E_2) +$

$\cos\frac{t}{\hbar}(E_1-E_2) - i\sin\frac{t}{\hbar}(E_1-E_2)] = a^2\Psi_1(x)^2 + b^2\Psi_2(x)^2 + 2ab\cos\frac{t}{\hbar}(E_1-E_2).$

(c) For which values of a and b is $\Psi(x,t)$ a stationary state of the system

$\Psi(x,t)$ stationary state of the system $\Rightarrow$ probability doesn't change with

time $\Rightarrow |\Psi(x,t)|^2$ is not a $f(t) \Rightarrow 2ab\cos\frac{t}{\hbar}(E_1-E_2) = 0$

$\Rightarrow a = 0 \quad$ o $\quad b = 0$

III.5 (15%) Let |nlm> refer to the (<u>orthonormal</u>) energy wavefunctions of the H atom ψ_{nlm}. Consider a system described by the following wavefunction Ψ:

$$\Psi = \frac{1}{\sqrt{5}}|300\rangle + \frac{1}{\sqrt{5}}|320\rangle + \frac{1}{\sqrt{5}}|310\rangle + \frac{1}{\sqrt{5}}|311\rangle + \frac{1}{\sqrt{5}}|31-1\rangle$$

i. What is the expectation value of the energy for this state? (expressed in terms of e^2/a_0)

$\langle E \rangle = E_3$: the eigenvalue . because the wavefunctions are degenerate

$$E_3 = \frac{-e^2}{2a_0 \cdot n^2} = \frac{-e^2}{2a_0 \cdot 9} = \frac{-e^2}{18 a_0}$$

ii. What is the expectation value of the orbital angular momentum for this state? $\hat{L}^2 \psi = \hbar^2 \ell(\ell+1) \psi$

$$\langle L^2 \rangle = \frac{1}{5} \cdot 0 + \frac{1}{5} \cdot \hbar^2 (2)(3) + \frac{1}{5} \cdot \hbar^2 (1)(2) + \frac{1}{5} \hbar^2 (1)(2) + \frac{1}{5} \hbar^2 (1)(2)$$

$$= \frac{\hbar^2}{5} \left(6 + 2 + 2 + 2 \right) = \frac{12 \hbar^2}{5} \quad :$$

iii. What is the expectation value of the z-component of the orbital angular momentum? $\hat{L}_z \psi = m\hbar \psi$

$$\langle L_z \rangle = \frac{1}{5} \cdot 0 + \frac{1}{5} \cdot 0 + \frac{1}{5} \cdot 0 + \frac{1}{5} \cdot (\hbar) + \frac{1}{5} (-\hbar) = 0$$

iv. What is the probability that a measurement of the orbital angular momentum is 0?

probability that $L=0$ then $L^2=0 \Rightarrow \ell=0$

$$\Rightarrow \frac{1}{5}$$

v. What is the probability that a measurement of the z-component of the orbital angular momentum is 0.

probability that $L_z=0 \Rightarrow m_\ell = 0$

$$\Rightarrow \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$$

a. $\hat{H} = -\dfrac{\hbar^2}{2m}\left(\dfrac{1}{r^2}\dfrac{\partial}{\partial r}r^2\dfrac{\partial}{\partial r} + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2}{\partial\phi^2}\right) + \dfrac{e^2}{r}$

$\hat{H}\,Y_\ell^{m_\ell}(\theta,\phi) = -\dfrac{\hbar^2}{2m}\left(\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2}{\partial\phi^2}\right)Y_\ell^m(\theta,\phi)$

$\qquad\qquad + \dfrac{e^2}{r}Y_\ell^{m_\ell}(\theta,\phi)$

$\qquad = \dfrac{\hat{L}^2}{2I}Y_\ell^m(\theta,\phi) + \dfrac{e^2}{r}Y_\ell^m(\theta,\phi)$

$\qquad = \dfrac{\hbar^2}{2I}\ell(\ell+1)Y_\ell^m + \dfrac{e^2}{r}Y_\ell^m$

$(r\text{ constant}) \qquad = \left(\dfrac{\hbar^2}{2I}\ell(\ell+1) + \dfrac{e^2}{r}\right)Y_\ell^m(\theta,\phi) \Rightarrow$ yes, an eigenfunction of $\hat{H}$.

b. $\hat{L}^2 Y_\ell^{m_\ell} = \hbar^2\ell(\ell+1)Y_\ell^{m_\ell}$ $\qquad\qquad \Rightarrow$ yes, an eigenfunction

c. $\hat{L}_z Y_\ell^{m_\ell} = m_\ell\hbar\, Y_\ell^{m_\ell}$ $\qquad\qquad \Rightarrow$ yes "

d. $\dfrac{\hat{L}^2}{2I}Y_\ell^{m_\ell} = \dfrac{\hbar^2\ell(\ell+1)}{2I}Y_\ell^m$